Worksheet for August 29

Problems marked with an asterisk are to be placed in your math diary.

(1*) For $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+2y^2}$, show that the limit along any line through the origin exists and equals zero, but if we take the limit along the curve $y = x^3$, the limit is not zero. What conclusion can you draw from this?

(2*) Determine if the following limit exists, and if so, what is its value: $\lim_{(x,y)\to(0,0)} \frac{x^5+y^4-3x^3y+2x^2+2y^2}{x^2+y^2}$.

(3*) The function $f(x,y) = 3x^2 + y$ is continuous, so that $\lim_{(x,y)\to(-1,3)} f(x,y) = 6$. Find δ, so that $||(x,y) - (-1,3)|| < \delta$ implies $|f(x,y) - 6| < 10^{-5}$. Can you find an optimal such δ, i.e., can you find δ satisfying the required relation for continuity, but if $||(x,y) - (-1,3)|| \geq \delta$, then $|f(x,y) - 6|$ is not less than 10^{-5}?

(4) Determine if the function $f(x,y) = \begin{cases} \frac{x^3+x^2+xy^2+y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0) \end{cases}$ is continuous at (0,0).

(5) At what values of (x,y) is the function $f(x,y) = \frac{\cos(x^2-y^2)}{x^2+1}$ continuous?

Optional Bonus Problem. 3 points, to be turned in with Tuesday's quiz. Use items (i) and (ii) below to prove that for $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$, $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, and then find its value.

(i) Prove that if a, b are real numbers, then $2|ab| \leq a^2 + b^2$.

(ii) Use (i) to show that if $0 < |(x,y)| < \delta$, then $|f(x,y)| < \frac{\delta^2}{2}$.